TD-04-017
May 20, 2004

Description of the Network Model for Rutherford-Type Cables

J. McDonald

Abstract:

A description of the electrical-network model for Rutherford-type cables is provided, including details of the solution method employed.

Problem description:

Consider a Rutherford cable of length L, containing N_s (assumed to be even) strands, and having a pitch length L_p. The cable length can be subdivided into N_b bands of length $L_b = L_p / N_s$, as pictured in Figure 1. Each band can then be decomposed into a collection of electrical sub-networks [1], as shown in Figure 2. Within each sub-network, the electrical contacts between strands are represented by ohmic resistors, and each section of strand is represented by a piece of wire with a resistance (either linear or nonlinear) and a self inductance; in addition, each strand section is linked to every other strand section in the cable by a mutual inductance.

The electrical network of each band contains $2N_s - 2$ strand sections, $2N_s$ intra-layer (adjacent) contact resistances, and $N_s - 1$ inter-layer (cross-over) contact resistances. Each strand section and contact resistance can carry a current; therefore, there are $5N_s - 3$ unknown currents per band, and $N_b \cdot (5N_s - 3)$ unknown currents in the entire cable.

To solve directly for all the unknown currents, each band must yield a system of $5N_s - 3$ linearly-independent equations. This system of equations can be derived by applying the Kirchhoff laws to loops and nodes within the network, and by constraining the net transport current through the cable cross section [1].

A more efficient method is to replace each contact current by the voltage difference at its end points (the contact nodes) divided by its resistance; this replacement allows the problem to be formulated in terms of the well-known transmission-line equations [2], and the resulting system can be solved directly for the currents in the strand sections. In this approach, the number of unknowns in the linear system is reduced to $N_b \cdot (2N_s - 2)$, but there is some extra post-processing involved in extracting the contact currents from the node voltages via the expressions in Equation (15).

Deriving the equations:

In symbolic form, the transmission-line equations can be expressed as [2]:

$$\begin{aligned} &\underline{\underline{\boldsymbol{M}}} \cdot \frac{d}{dt} \underline{\boldsymbol{i}}_s + \underline{\underline{\boldsymbol{R}}}_s \cdot \underline{\boldsymbol{i}}_s = \underline{\boldsymbol{\varepsilon}}_s^{ext} - \underline{\boldsymbol{\delta}\ \boldsymbol{U}}_s \\ &\underline{\underline{\boldsymbol{G}}} \cdot \underline{\boldsymbol{U}} = \underline{\boldsymbol{\delta}\ \boldsymbol{i}}_s \end{aligned} \tag{1}$$

where $\underline{\boldsymbol{i}}_s$ is the vector of currents in the strand sections, $\underline{\boldsymbol{U}}$ is the vector of voltages at the contact nodes, $\underline{\boldsymbol{\delta}\ \boldsymbol{i}}_s$ is the vector of current changes along the strand sections (due to inter-strand current transfer), $\underline{\boldsymbol{\delta}\ \boldsymbol{U}}_s$ is the vector of voltage changes along the strand sections, $\underline{\underline{\boldsymbol{M}}}$ is the inductance matrix, $\underline{\underline{\boldsymbol{R}}}_s$ is the diagonal matrix containing the resistances of the strand sections, $\underline{\underline{\boldsymbol{G}}}$ is the conductance matrix describing the electrical contacts between the strands, and $\underline{\boldsymbol{\varepsilon}}_s^{ext}$ is the vector of externally-induced electro-motive forces along the strand sections.

To obtain the explicit form of Equation (1) we need to apply the Kirchhoff laws to our particular electrical network. For band n, let the strand currents be denoted by $i_{s,\ n,\ j}\quad j = 2,\ldots,\ 2N_s - 2$, the cross-over currents by $i_{c,\ n,\ j}\quad j = 1,\ldots,\ N_s - 1$, and the adjacent currents by $i_{a,\ n,\ j}\quad j = 1,\ldots,\ 2N_s$; the contact resistances are labeled using the same notation (see Figs.). Let the individual sub-networks in each band be labeled by an index $v = 2,\ldots,\ \frac{1}{2}N_s + 1$; and let $\dot{\Phi}^{ext}_{n,\ v,\ f}$ denote the time-rate-of-change of the outward-directed external flux for face f of sub-network v, as depicted in Figures 2-5. The inductive-matrix element linking strand section j in band n and strand section k in band m is denoted by $M_{j,n}^{k,m}$. Using the specified notation, the first part of Equation (1) takes the form:

$$\begin{aligned}
&\sum_{m,k} M_{n,2}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ 2}\ i_{s,\ n,\ 2} = \varepsilon_{s,\ n,\ 2}^{ext} - \delta\ U_{s,\ n,\ 2} \\
&\sum_{m,k} M_{n,3}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ 3}\ i_{s,\ n,\ 3} = \varepsilon_{s,\ n,\ 3}^{ext} - \delta\ U_{s,\ n,\ 3}
\end{aligned} \tag{2}$$

$$\begin{aligned}
&v = 2, \ldots,\ \frac{1}{2} N_s - 1 \\
&\sum_{m,k} M_{n,2v}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ 2v}\ i_{s,\ n,\ 2v} = \varepsilon_{s,\ n,\ 2v}^{ext} - \delta\ U_{s,\ n,\ 2v} \\
&\sum_{m,k} M_{n,2v+1}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ 2v+1}\ i_{s,\ n,\ 2v+1} = \varepsilon_{s,\ n,\ 2v+1}^{ext} - \delta\ U_{s,\ n,\ 2v+1}
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\sum_{m,k} M_{n,N_S}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ N_S}\ i_{s,\ n,\ N_S} = \varepsilon_{s,\ n,\ N_S}^{ext} - \delta\ U_{s,\ n,\ N_S} \\
&\sum_{m,k} M_{n,N_S+1}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ N_S+1}\ i_{s,\ n,\ N_S+1} = \varepsilon_{s,\ n,\ N_S+1}^{ext} - \delta\ U_{s,\ n,\ N_S+1}
\end{aligned} \tag{4}$$

$$\begin{aligned}
&v = 2, \ldots,\ \frac{1}{2} N_s - 1 \\
&\sum_{m,k} M_{n,N_S+2v-2}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ N_S+2v-2}\ i_{s,\ n,\ N_S+2v-2} = \varepsilon_{s,\ n,\ N_S+2v-2}^{ext} - \delta\ U_{s,\ n,\ N_S+2v-2} \\
&\sum_{m,k} M_{n,N_S+2v-1}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ N_S+2v-1}\ i_{s,\ n,\ N_S+2v-1} = \varepsilon_{s,\ n,\ N_S+2v-1}^{ext} - \delta\ U_{s,\ n,\ N_S+2v-1}
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\sum_{m,k} M_{n,2N_S-2}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ 2N_S-2}\ i_{s,\ n,\ 2N_S-2} = \varepsilon_{s,\ n,\ 2N_S-2}^{ext} - \delta\ U_{s,\ n,\ 2N_S-2} \\
&\sum_{m,k} M_{n,2N_S-1}^{m,k} \frac{d}{dt} i_{s,\ m,\ k} + r_{s,\ n,\ 2N_S-1}\ i_{s,\ n,\ 2N_S-1} = \varepsilon_{s,\ n,\ 2N_S-1}^{ext} - \delta\ U_{s,\ n,\ 2N_S-1}
\end{aligned} \tag{6}$$

with the identifications:

$$
\begin{aligned}
&\delta\, U_{s,\; n,\; 2} = U_{n+1,\; 1} - U_{n,\; 2} \\
&\delta\, U_{s,\; n,\; 3} = U_{n,\; 2} - U_{n,\; 4} \\
&\left.\begin{aligned}
&\delta\, U_{s,\; n,\; 2v} = U_{n,\; 4v-3} - U_{n,\; 4v-5} \\
&\delta\, U_{s,\; n,\; 2v+1} = U_{n,\; 4v-2} - U_{n,\; 4v}
\end{aligned}\right\} \; v = 2, \ldots,\; \frac{1}{2} N_s - 1 \\
&\delta\, U_{s,\; n,\; N_S} = U_{n,\; 2N_S-3} - U_{n,\; 2N_S-5} \\
&\delta\, U_{s,\; n,\; N_S+1} = U_{n+1,\; 3} - U_{n,\; 1} \\
&\left.\begin{aligned}
&\delta\, U_{s,\; n,\; N_S+2v-2} = U_{n+1,\; 4v-4} - U_{n,\; 4v-2} \\
&\delta\, U_{s,\; n,\; N_S+2v-1} = U_{n+1,\; 4v-1} - U_{n,\; 4v-3}
\end{aligned}\right\} \; v = 2, \ldots,\; \frac{1}{2} N_s - 1 \\
&\delta\, U_{s,\; n,\; 2N_S-2} = U_{n+1,\; 2N_S-4} - U_{n,\; 2N_S-2} \\
&\delta\, U_{s,\; n,\; 2N_S-1} = U_{n+1,\; 2N_S-2} - U_{n,\; 2N_S-3}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\varepsilon^{ext}_{s,\; n,\; 2} = \dot{\Phi}^{ext}_{n,\; 1,\; 2} - \dot{\Phi}^{ext}_{n,\; 1,\; 3} \\
&\varepsilon^{ext}_{s,\; n,\; 3} = \dot{\Phi}^{ext}_{n-1,\; 1,\; 4} + \dot{\Phi}^{ext}_{n,\; 2,\; 1} - \dot{\Phi}^{ext}_{n,\; 2,\; 6} \\
&\left.\begin{aligned}
&\varepsilon^{ext}_{s,\; n,\; 2v} = \dot{\Phi}^{ext}_{n-1,\; v,\; 7} + \dot{\Phi}^{ext}_{n,\; v,\; 2} - \dot{\Phi}^{ext}_{n,\; v,\; 5} \\
&\varepsilon^{ext}_{s,\; n,\; 2v+1} = \dot{\Phi}^{ext}_{n-1,\; v,\; 8} + \dot{\Phi}^{ext}_{n,\; v+1,\; 1} - \dot{\Phi}^{ext}_{n,\; v+1,\; 6}
\end{aligned}\right\} \; v = 2, \ldots,\; \frac{1}{2} N_s - 1 \\
&\varepsilon^{ext}_{s,\; n,\; N_S} = \dot{\Phi}^{ext}_{n-1,\; N_S/2+1,\; 1} + \dot{\Phi}^{ext}_{n,\; N_S/2,\; 2} - \dot{\Phi}^{ext}_{n,\; N_S/2,\; 5} \\
&\varepsilon^{ext}_{s,\; n,\; N_S+1} = -\dot{\Phi}^{ext}_{n,\; 1,\; 1} - \dot{\Phi}^{ext}_{n,\; 2,\; 4} + \dot{\Phi}^{ext}_{n,\; 2,\; 5} \\
&\left.\begin{aligned}
&\varepsilon^{ext}_{s,\; n,\; N_S+2v-2} = -\dot{\Phi}^{ext}_{n,\; v,\; 3} + \dot{\Phi}^{ext}_{n,\; v,\; 6} - \dot{\Phi}^{ext}_{n,\; v,\; 8} \\
&\varepsilon^{ext}_{s,\; n,\; N_S+2v-1} = -\dot{\Phi}^{ext}_{n,\; v+1,\; 4} + \dot{\Phi}^{ext}_{n,\; v+1,\; 5} - \dot{\Phi}^{ext}_{n,\; v,\; 7}
\end{aligned}\right\} \; v = 2, \ldots,\; \frac{1}{2} N_s - 1 \\
&\varepsilon^{ext}_{s,\; n,\; 2N_S-2} = -\dot{\Phi}^{ext}_{n,\; N_S/2,\; 3} + \dot{\Phi}^{ext}_{n,\; N_S/2,\; 6} - \dot{\Phi}^{ext}_{n,\; N_S/2+1,\; 4} \\
&\varepsilon^{ext}_{s,\; n,\; 2N_S-1} = -\dot{\Phi}^{ext}_{n,\; N_S/2+1,\; 2} + \dot{\Phi}^{ext}_{n,\; N_S/2+1,\; 3}
\end{aligned}
\tag{8}
$$

To represent the second part of Equation (1), we must consider current conservation in the contact nodes, as depicted in Figure 6:

$$- i_{a,\ n-1,\ 1} + i_{s,\ n,\ 2} - i_{s,\ n,\ 3} - i_{c,\ n,\ 1} + i_{a,\ n,\ 1} + i_{a,\ n,\ N_S+1} = 0 \tag{9}$$

$$- i_{s,\ n-1,\ 2} - i_{a,\ n-1,\ 2} + i_{s,\ n,\ N_S+1} + i_{c,\ n,\ 1} + i_{a,\ n,\ 2} - i_{a,\ n,\ 3} = 0 \tag{10}$$

$$\left.\begin{aligned} - i_{s,\ n-1,\ N_S+2v-2} - i_{a,\ n-1,\ N_S+2v-3} + i_{s,\ n,\ 2v-1} - i_{c,\ n,\ 2v-2} + i_{a,\ n,\ 2v} &= 0 \\ - i_{s,\ n-1,\ N_S+2v-3} - i_{a,\ n-1,\ N_S+2v-2} + i_{s,\ n,\ 2v} + i_{c,\ n,\ 2v-2} + i_{a,\ n,\ 2v-1} &= 0 \end{aligned}\right\} v = 2,\ldots,\frac{1}{2}N_s \tag{11}$$

$$\left.\begin{aligned} - i_{s,\ n,\ 2v+1} + i_{s,\ n,\ N_S+2v-2} - i_{c,\ n,\ 2v-1} - i_{a,\ n,\ 2v} + i_{a,\ n,\ N_S+2v-1} &= 0 \\ - i_{s,\ n,\ 2v} + i_{s,\ n,\ N_S+2v-1} + i_{c,\ n,\ 2v-1} - i_{a,\ n,\ 2v+1} + i_{a,\ n,\ N_S+2v-2} &= 0 \end{aligned}\right\} v = 2,\ldots,\frac{1}{2}N_s - 1 \tag{12}$$

$$- i_{s,\ n-1,\ 2N_S-1} - i_{a,\ n-1,\ 2N_S} + i_{s,\ n,\ 2N_S-2} - i_{c,\ n,\ N_S-1} - i_{a,\ n,\ N_S} + i_{a,\ n,\ 2N_S} = 0 \tag{13}$$

$$- i_{a,\ n-1,\ 2N_S-1} - i_{s,\ n,\ N_S} + i_{s,\ n,\ 2N_S-1} + i_{c,\ n,\ N_S-1} + i_{a,\ n,\ 2N_S-2} + i_{a,\ n,\ 2N_S-1} = 0 \tag{14}$$

The contact currents can be eliminated from Equations (9)-(14) using the relations:

$$\begin{aligned}
& r_{c,\ n,\ 1}\, i_{c,\ n,\ 1} = U_{n,\ 1} - U_{n,\ 2} \\
& \left.\begin{aligned} r_{c,\ n,\ 2v-2}\, i_{c,\ n,\ 2v-2} &= U_{n,\ 4v-5} - U_{n,\ 4v-4} \\ r_{c,\ n,\ 2v-1}\, i_{c,\ n,\ 2v-1} &= U_{n,\ 4v-3} - U_{n,\ 4v-2} \end{aligned}\right\} v = 2,\ldots,\ \frac{1}{2}N_s \\
& r_{a,\ n,\ 1}\, i_{a,\ n,\ 1} = U_{n,\ 2} - U_{n+1,\ 2} \\
& r_{a,\ n,\ 2}\, i_{a,\ n,\ 2} = U_{n,\ 1} - U_{n+1,\ 1} \\
& \left.\begin{aligned} r_{a,\ n,\ 2v-1}\, i_{a,\ n,\ 2v-1} &= U_{n,\ 4v-5} - U_{n,\ 4v-7} \\ r_{a,\ n,\ 2v}\, i_{a,\ n,\ 2v} &= U_{n,\ 4v-4} - U_{n,\ 4v-2} \\ r_{a,\ n,\ N_S+2v-3}\, i_{a,\ n,\ N_S+2v-3} &= U_{n,\ 4v-6} - U_{n+1,\ 4v-4} \\ r_{a,\ n,\ N_S+2v-2}\, i_{a,\ n,\ N_S+2v-2} &= U_{n,\ 4v-3} - U_{n+1,\ 4v-5} \end{aligned}\right\} v = 2,\ldots,\ \frac{1}{2}N_s \\
& r_{a,\ n,\ 2N_S-1}\, i_{a,\ n,\ 2N_S-1} = U_{n,\ 2N_S-3} - U_{n+1,\ 2N_S-3} \\
& r_{a,\ n,\ 2N_S}\, i_{a,\ n,\ 2N_S} = U_{n,\ 2N_S-2} - U_{n+1,\ 2N_S-2}
\end{aligned} \tag{15}$$

Combining Equations (9)-(15) with the definitions:

$$
\begin{aligned}
&\delta\, i_{s,\,n,\,2} = i_{s,\,n,\,2} - i_{s,\,n,\,3} \\
&\left.\begin{aligned}
\delta\, i_{s,\,n,\,2v-1} &= i_{s,\,n,\,2v-1} - i_{s,\,n-1,\,N_S+2v-2} \\
\delta\, i_{s,\,n,\,2v} &= i_{s,\,n,\,2v} - i_{s,\,n-1,\,N_S+2v-3}
\end{aligned}\right\} \; v = 2,\ldots,\ \frac{1}{2}N_s \\
&\delta\, i_{s,\,n,\,N_S+1} = i_{s,\,n,\,N_S+1} - i_{s,\,n-1,\,2} \\
&\left.\begin{aligned}
\delta\, i_{s,\,n,\,N_S+2v-2} &= i_{s,\,n,\,N_S+2v-2} - i_{s,\,n,\,2v+1} \\
\delta\, i_{s,\,n,\,N_S+2v-1} &= i_{s,\,n,\,N_S+2v-1} - i_{s,\,n,\,2v}
\end{aligned}\right\} \; v = 2,\ldots,\ \frac{1}{2}N_s - 1 \\
&\delta\, i_{s,\,n,\,2N_S-2} = i_{s,\,n,\,2N_S-2} - i_{s,\,n-1,\,2N_S-1} \\
&\delta\, i_{s,\,n,\,2N_S-1} = i_{s,\,n,\,2N_S-1} - i_{s,\,n,\,N_S}
\end{aligned}
\tag{16}
$$

yields the expressions:

$$
\begin{aligned}
\delta\, i_{s,\,n,\,2} = {}& \frac{U_{n-1,\,2}}{r_{a,\,n-1,\,1}} + \frac{U_{n,\,1}}{r_{c,\,n,\,1}} - \left(\frac{1}{r_{a,\,n-1,\,1}} + \frac{1}{r_{c,\,n,\,1}} + \frac{1}{r_{a,\,n,\,1}} + \frac{1}{r_{a,\,n,\,N_S+1}} \right) U_{n,\,2} \\
& + \frac{U_{n+1,\,2}}{r_{a,\,n,\,1}} + \frac{U_{n+1,\,4}}{r_{a,\,n,\,N_S+1}}
\end{aligned}
\tag{17}
$$

$$
\left.
\begin{aligned}
\delta\, i_{s,\,n,\,2v-1} = {}& \frac{U_{n-1,\,4v-6}}{r_{a,\,n-1,\,N_S+2v-3}} + \frac{U_{n,\,4v-5}}{r_{c,\,n,\,2v-2}} \\
& - \left(\frac{1}{r_{a,\,n-1,\,N_S+2v-3}} + \frac{1}{r_{c,\,n,\,2v-2}} + \frac{1}{r_{a,\,n,\,2v}} \right) U_{n,\,4v-4} + \frac{U_{n,\,4v-2}}{r_{a,\,n,\,2v}} \\
\delta\, i_{s,\,n,\,2v} = {}& \frac{U_{n-1,\,4v-3}}{r_{a,\,n-1,\,N_S+2v-2}} + \frac{U_{n,\,4v-7}}{r_{a,\,n,\,2v-1}} \\
& - \left(\frac{1}{r_{a,\,n-1,\,N_S+2v-2}} + \frac{1}{r_{a,\,n,\,2v-1}} + \frac{1}{r_{c,\,n,\,2v-2}} \right) U_{n,\,4v-5} + \frac{U_{n,\,4v-4}}{r_{c,\,n,\,2v-2}}
\end{aligned}
\right\} \; v = 2,\ldots,\ \frac{1}{2}N_s
\tag{18}
$$

$$\delta\ i_{s,\ n,\ N_S+1}=\frac{U_{n-1,\ 1}}{r_{a,\ n-1,\ 2}}-\left(\frac{1}{r_{a,\ n-1,\ 2}}+\frac{1}{r_{c,\ n,\ 1}}+\frac{1}{r_{a,\ n,\ 3}}+\frac{1}{r_{a,\ n,\ 2}}\right)U_{n,\ 1}$$
$$+\frac{U_{n,\ 2}}{r_{c,\ n,\ 1}}+\frac{U_{n,\ 3}}{r_{a,\ n,\ 3}}+\frac{U_{n+1,\ 1}}{r_{a,\ n,\ 2}} \qquad (19)$$

$$\left.\begin{aligned}\delta\ i_{s,\ n,\ N_S+2v-2}=&\frac{U_{n,\ 4v-4}}{r_{a,\ n,\ 2v}}+\frac{U_{n,\ 4v-3}}{r_{c,\ n,\ 2v-1}}\\&-\left(\frac{1}{r_{a,\ n,\ 2v}}+\frac{1}{r_{c,\ n,\ 2v-1}}+\frac{1}{r_{a,\ n,\ N_S+2v-1}}\right)U_{n,\ 4v-2}+\frac{U_{n+1,\ 4v}}{r_{a,\ n,\ N_S+2v-1}}\\\delta\ i_{s,\ n,\ N_S+2v-1}=&-\left(\frac{1}{r_{c,\ n,\ 2v-1}}+\frac{1}{r_{a,\ n,\ 2v+1}}+\frac{1}{r_{a,\ n,\ N_S+2v-2}}\right)U_{n,\ 4v-3}\\&+\frac{U_{n,\ 4v-2}}{r_{c,\ n,\ 2v-1}}+\frac{U_{n,\ 4v-1}}{r_{a,\ n,\ 2v+1}}+\frac{U_{n+1,\ 4v-5}}{r_{a,\ n,\ N_S+2v-2}}\end{aligned}\right\}\ v=2,\ldots,\ \frac{1}{2}N_s-1$$

$$(20)$$

$$\delta\ i_{s,\ n,\ 2N_S-2}=\frac{U_{n-1,\ 2N_S-2}}{r_{a,\ n-1,\ 2N_S}}+\frac{U_{n,\ 2N_S-4}}{r_{a,\ n,\ N_S}}+\frac{U_{n,\ 2N_S-3}}{r_{c,\ n,\ N_S-1}}$$
$$-\left(\frac{1}{r_{a,\ n-1,\ 2N_S}}+\frac{1}{r_{a,\ n,\ N_S}}+\frac{1}{r_{c,\ n,\ N_S-1}}+\frac{1}{r_{a,\ n,\ 2N_S}}\right)U_{n,\ 2N_S-2}+\frac{U_{n+1,\ 2N_S-2}}{r_{a,\ n,\ 2N_S}} \qquad (21)$$

$$\delta\ i_{s,\ n,\ 2N_S-1}=\frac{U_{n-1,\ 2N_S-3}}{r_{a,\ n-1,\ 2N_S-1}}$$
$$-\left(\frac{1}{r_{a,\ n-1,\ 2N_S-1}}+\frac{1}{r_{c,\ n,\ N_S-1}}+\frac{1}{r_{a,\ n,\ 2N_S-2}}+\frac{1}{r_{a,\ n,\ 2N_S-1}}\right)U_{n,\ 2N_S-3} \qquad (22)$$
$$+\frac{U_{n,\ 2N_S-2}}{r_{c,\ n,\ N_S-1}}+\frac{U_{n+1,\ 2N_S-5}}{r_{a,\ n,\ 2N_S-2}}+\frac{U_{n+1,\ 2N_S-3}}{r_{a,\ n,\ 2N_S-1}}$$

Each of the Equations in (17)-(22) defines a single row of the matrix $\underline{\underline{\boldsymbol{G}}}$.

Solving the equations:

In order to obtain a unique solution, we need to impose boundary conditions at the ends of the cable. We will use fixed-current conditions defined by:

$$\begin{aligned} i_{s,\ 0,\ j} &= \frac{I_T}{N_s},\quad j = 2,\ \ N_s + 1, \ldots,\ 2N_s - 1 \\ i_{a,\ 0,\ j} &= 0 \quad,\quad j = 1,\ 2,\ \ N_s + 1, \ldots,\ 2N_s \\ i_{s,\ N_b,\ j} &= \frac{I_T}{N_s},\quad j = 2,\ \ N_s + 1, \ldots,\ 2N_s - 1 \\ i_{a,\ N_b,\ j} &= 0 \quad,\quad j = 1,\ 2,\ \ N_s + 1, \ldots,\ 2N_s \end{aligned} \tag{23}$$

where I_T is the transport current in the cable.

Equation (1) can be integrated forward in time using an explicit Euler scheme [3, 4]:

$$\begin{aligned} &given\ \ \underline{\boldsymbol{i}}_s^{(0)},\ \underline{\boldsymbol{U}}^{(0)} \\ &for\ \ q = 1, \ldots \\ &\qquad t = q \cdot \Delta\ t \\ &\qquad \left(\underline{\underline{\boldsymbol{M}}} + \Delta\ t \cdot \underline{\underline{\boldsymbol{R}}}_s\right) \cdot \underline{\boldsymbol{i}}_s^{(q)} = \underline{\underline{\boldsymbol{M}}} \cdot \underline{\boldsymbol{i}}_s^{(q-1)} + \Delta\ t \cdot \left(\underline{\boldsymbol{\varepsilon}}_s^{ext} - \underline{\boldsymbol{\delta}\ \boldsymbol{U}}_s\right)^{(q-1)} \\ &\qquad \underline{\boldsymbol{U}}^{(q)} = \underline{\underline{\boldsymbol{G}}}^{-1} \cdot \underline{\boldsymbol{\delta}\ \boldsymbol{i}}_s^{(q)} \\ &end\ \ for \end{aligned} \tag{24}$$

When the matrix $\underline{\underline{\boldsymbol{R}}}_s$ is a function of the vector $\underline{\boldsymbol{i}}_s$, for example in the case a power-law strand resistivity, then some form of iteration must be performed at each time step. The damped Newton iteration is generally regarded as the best approach [5]. Using the definitions: $\underline{\underline{\boldsymbol{A}}} \equiv \underline{\underline{\boldsymbol{M}}} + \Delta\ t \cdot \underline{\underline{\boldsymbol{R}}}_s$, $\underline{\boldsymbol{x}} \equiv \underline{\boldsymbol{i}}_s^{(q)}$, $\underline{\boldsymbol{x}}_{(0)} \equiv \underline{\boldsymbol{i}}_s^{(q-1)}$, and

$\underline{\boldsymbol{b}} \equiv \underline{\underline{\boldsymbol{M}}} \cdot \underline{\boldsymbol{i}}_s^{(q-1)} + \Delta\ t \cdot \left(\underline{\boldsymbol{\varepsilon}}_s^{ext} - \underline{\boldsymbol{\delta}\ \boldsymbol{U}}_s\right)^{(q-1)}$, the damped Newton iteration takes the form:

$$
\begin{aligned}
& given \quad \underline{\boldsymbol{x}}_{(0)} \;,\; \varepsilon << 1 \\
& \underline{\boldsymbol{r}}_{(0)} = \underline{\underline{\boldsymbol{A}}}\left(\underline{\boldsymbol{x}}_{(0)}\right) \cdot \underline{\boldsymbol{x}}_{(0)} - \underline{\boldsymbol{b}} \\
& for \quad p = 1, \ldots \\
& \qquad \underline{\underline{\boldsymbol{J}}}_{(p-1)} = \underline{\underline{\boldsymbol{A}}}\left(\underline{\boldsymbol{x}}_{(p-1)}\right) - \left(\frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial \underline{\boldsymbol{x}}}\right)_{(p-1)} \cdot \underline{\boldsymbol{x}}_{(p-1)} \\
& \qquad \underline{\underline{\boldsymbol{J}}}_{(p-1)} \cdot \Delta \underline{\boldsymbol{x}}_{(p)} = -\underline{\boldsymbol{r}}_{(p-1)} \\
& \qquad for \quad m = 0, \ldots \\
& \qquad\qquad \underline{\boldsymbol{x}}_{(p)} = \underline{\boldsymbol{x}}_{(p-1)} + \left(\frac{1}{2}\right)^{m} \cdot \Delta \underline{\boldsymbol{x}}_{(p)} \\
& \qquad\qquad \underline{\boldsymbol{r}}_{(p)} = \underline{\underline{\boldsymbol{A}}}\left(\underline{\boldsymbol{x}}_{(p)}\right) \cdot \underline{\boldsymbol{x}}_{(p)} - \underline{\boldsymbol{b}} \\
& \qquad\qquad if \quad \left| \; \underline{\boldsymbol{r}}_{(p)} \; \right| < \left| \; \underline{\boldsymbol{r}}_{(p-1)} \; \right| \quad exit \quad for \\
& \qquad end \quad for \\
& \qquad if \quad \left| \; \underline{\boldsymbol{r}}_{(p)} \; \right| < \varepsilon \left| \; \underline{\boldsymbol{b}} \; \right| \quad exit \quad for \\
& end \quad for
\end{aligned}
\tag{25}
$$

Within a single non-linear iteration we need to solve a linear system of the form $\underline{\underline{\boldsymbol{J}}}_{(p-1)} \cdot \Delta \underline{\boldsymbol{x}}_{(p)} = -\underline{\boldsymbol{r}}_{(p-1)}$. Because the matrix $\underline{\underline{\boldsymbol{J}}}_{(p-1)}$ is not sparse, a direct solution becomes prohibitively expensive for large cables. Fortunately, because the matrix $\underline{\underline{\boldsymbol{J}}}_{(p-1)}$ is symmetric and positive definite, we can use the well-known conjugate-gradients method [4, 6], in which the solution is projected onto a subspace of orthogonal vectors, constructed using matrix-vector multiplications. For a non-sparse matrix, however, each matrix-vector product can become very costly, both in operation count and in storage requirements. Luckily, the complexity of the matrix-vector products can be reduced by an order of magnitude by implementing a modern algorithm called the Fast-Multipole Method [7-11].

References:

[1] A. P. Verweij, PhD Thesis, University of Twente, Enschede, Netherlands (1995).

[2] L. Bottura, M. Breschi, and M. Fabbri, *J. Appl. Phys*. **92**, pp. 7571-7580 (2002).

[3] W. F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Academic Press, New York, NY (1965).

[4] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes*, 2nd ed., Cambridge University Press, New York, NY (1992).

[5] G. Engeln-Müllges and F. Uhlig, *Numerical Algorithms with Fortran*, Springer-Verlag, New York, NY (1996).

[6] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed. (2000)

[7] J. McDonald, Fermilab Technical Document: TD-04-018 (2004).

[8] V. Rokhlin, *J. Comput. Phys.* **60**, pp. 187-207 (1985).

[9] L. Greengard, *Wavelets, Multilevel Methods and Elliptic PDEs*, M. Ainsworth, J. Levesley, W. A. Light, and M. Marletta eds., Oxford University Press, Oxford, England (1997).

[10] C. Seberino and H. N. Bertram, *IEEE Trans. Mag.*, vol. 37, pp. 1078-1086 (2001).

[11] G. Rubinacci, A. Tamburrino, S. Ventre, and F. Villone, *IEEE Trans. Mag.*, vol. 38, pp. 529-532 (2002)

Figures:

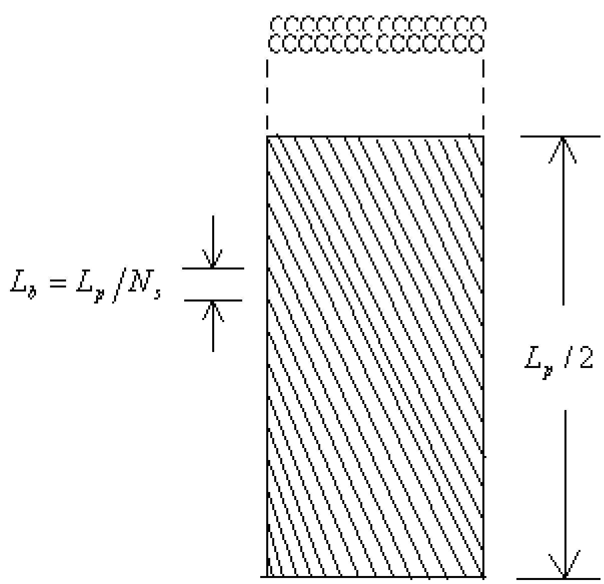

Figure 1 Half of a pitch length for a 28-strand cable.

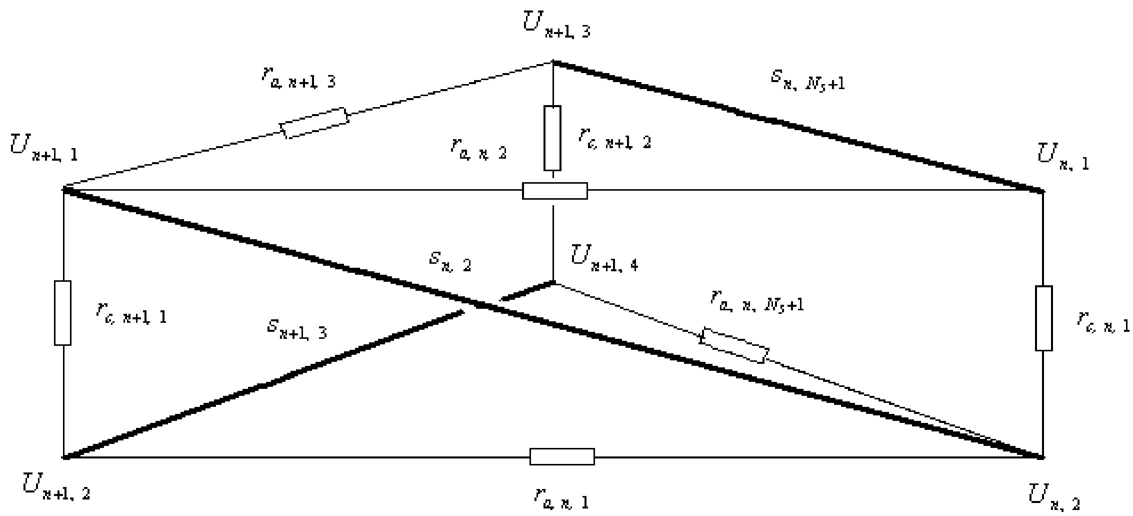

Figure 2(a) Sub-network $v = 1$ in band n.

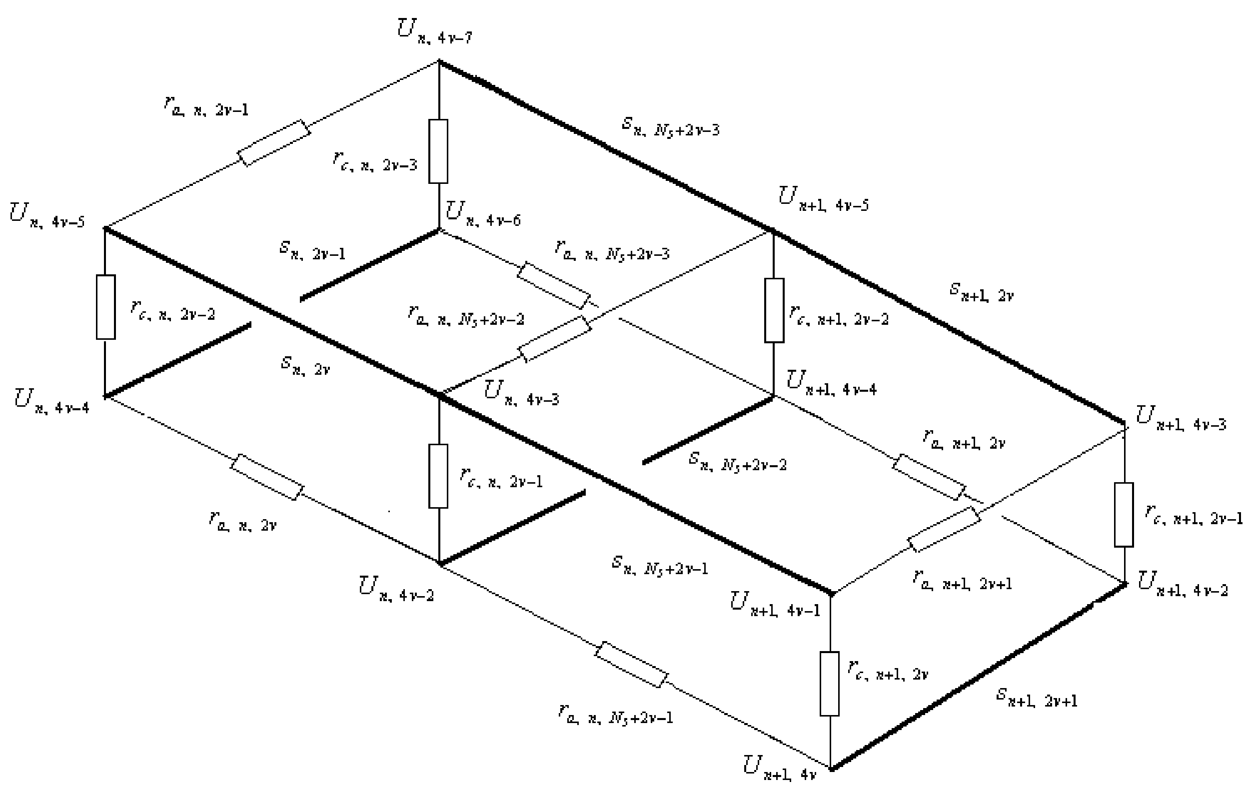

Figure 2(b) Sub-networks $v = 2, \ldots, \frac{1}{2}N_s - 1$ in band n.

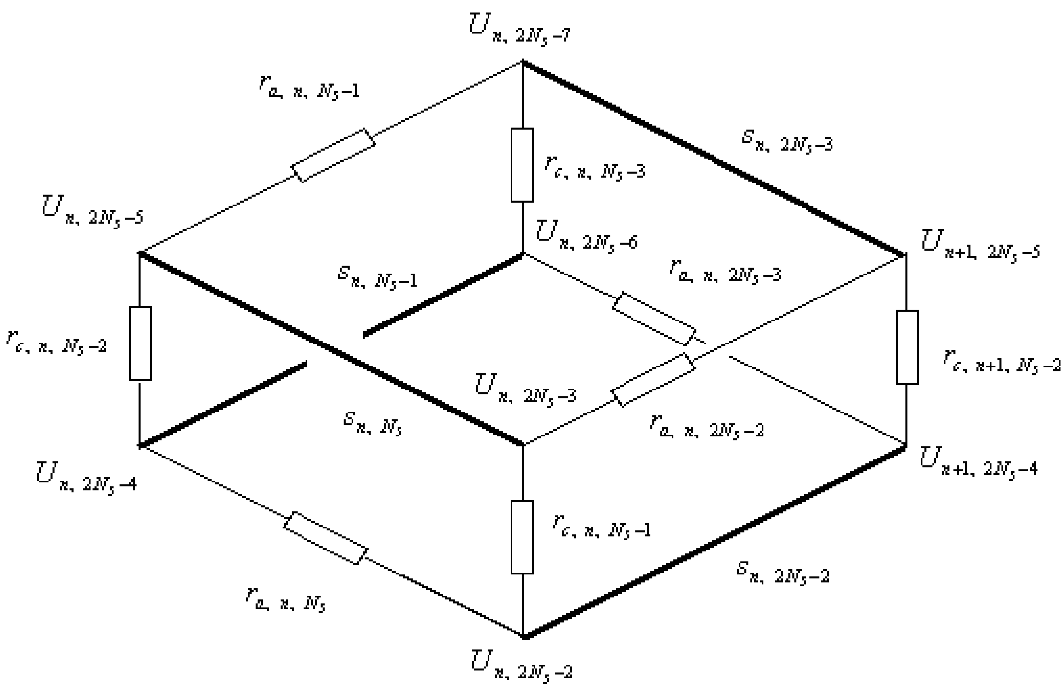

Figure 2(c) Sub-network $v = \frac{1}{2}N_s$ in band n.

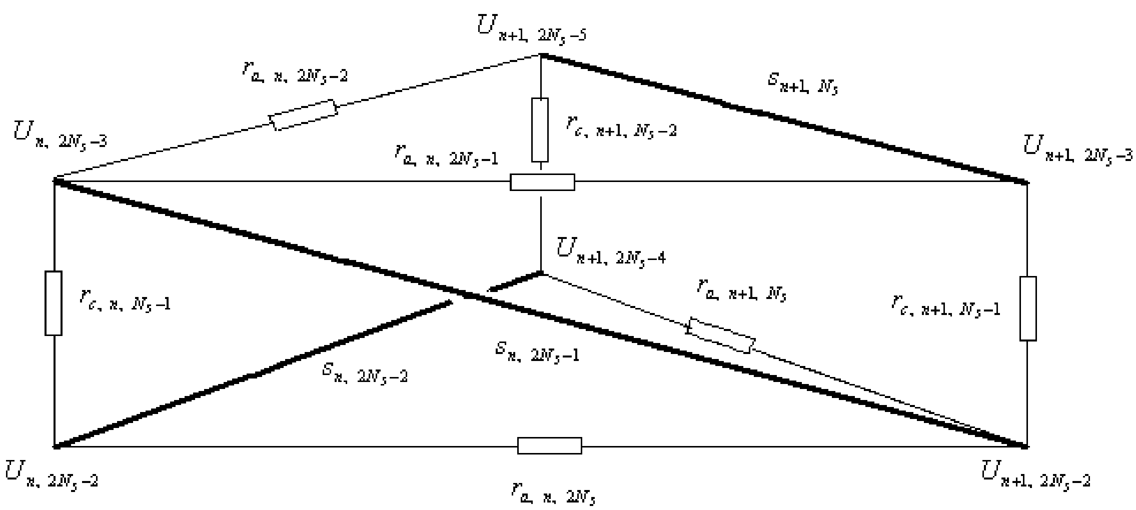

Figure 2(d) Sub-network $v = \frac{1}{2}N_s + 1$ in band n.

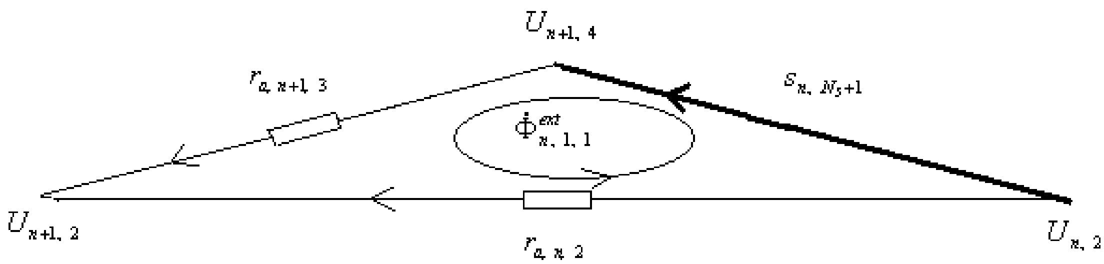

Figure 3(a) Face $f=1$ of sub-network $v=1$ in band n.

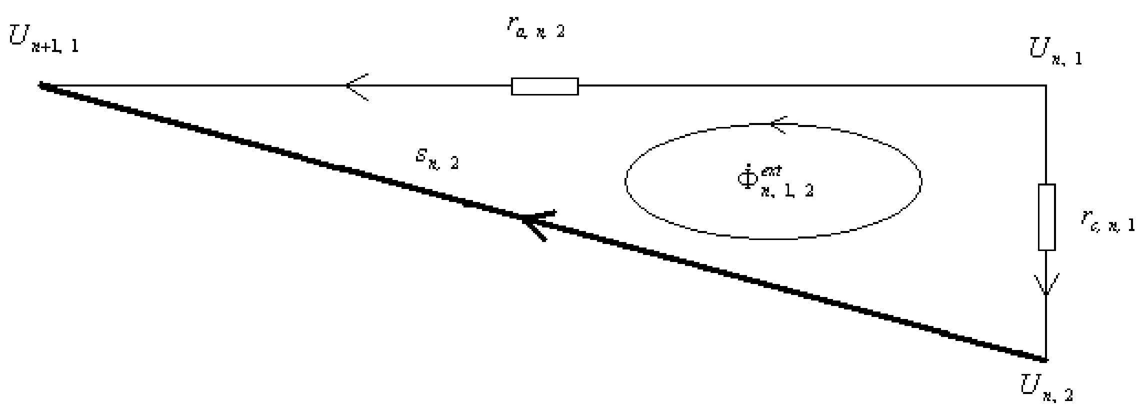

Figure 3(b) Face $f=2$ of sub-network $v=1$ in band n.

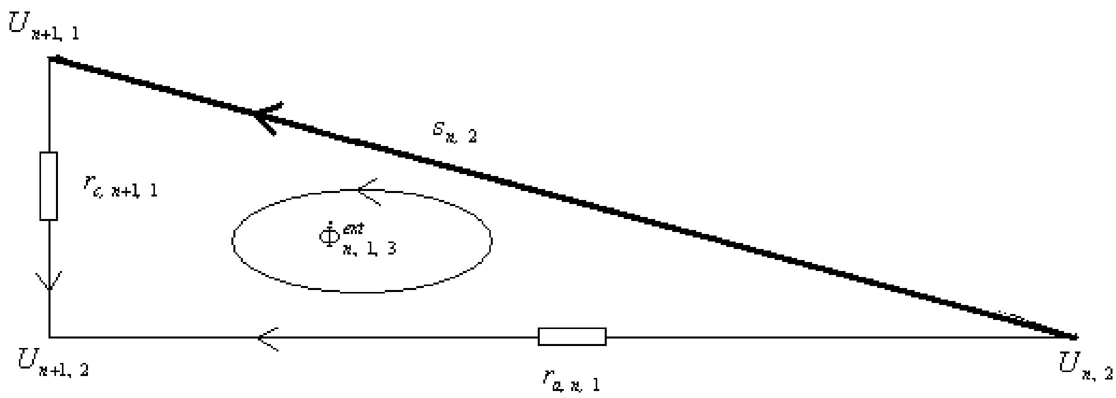

Figure 3(c) Face $f = 3$ of sub-network $v = 1$ in band n.

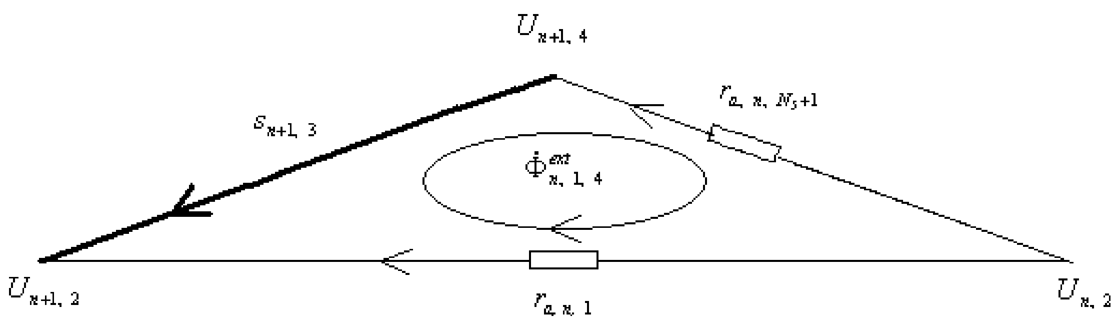

Figure 3(d) Face $f = 4$ of sub-network $v = 1$ in band n.

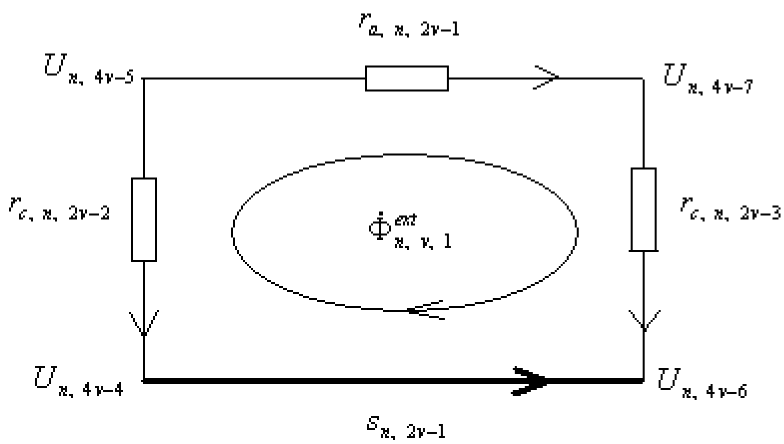

Figure 4(a) Face $f = 1$ of sub-networks $v = 2, \ldots, \frac{1}{2}N_s - 1$ in band n.

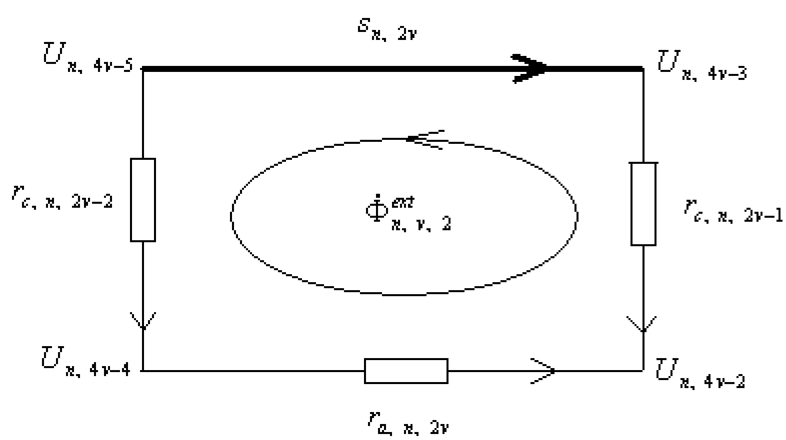

Figure 4(b) Face $f = 2$ of sub-networks $v = 2, \ldots, \frac{1}{2}N_s - 1$ in band n.

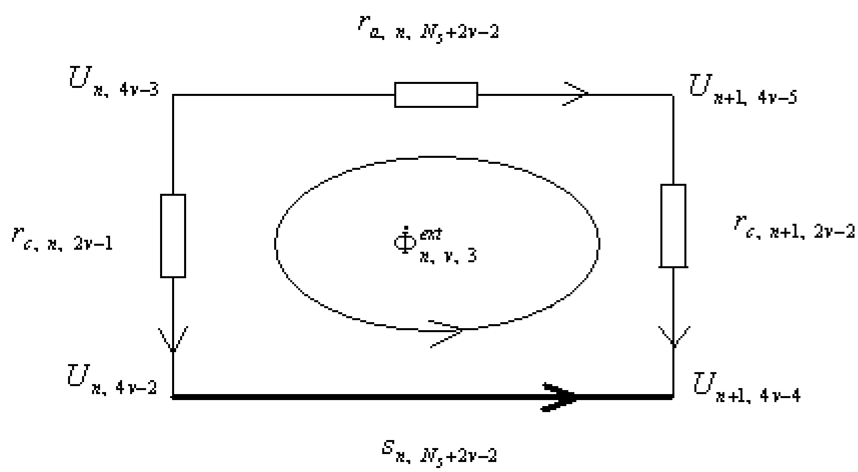

Figure 4(c)   Face $f = 3$ of sub-networks $v = 2, \ldots, \frac{1}{2}N_s - 1$ in band n.

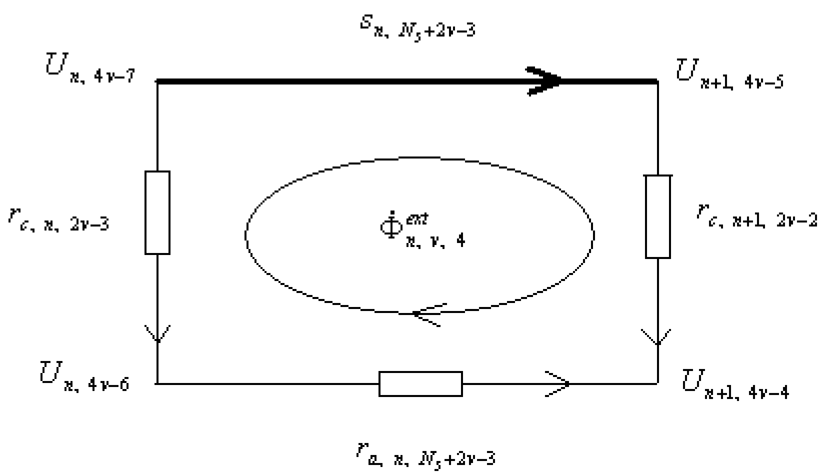

Figure 4(d)   Face $f = 4$ of sub-networks $v = 2, \ldots, \frac{1}{2}N_s - 1$ in band n.

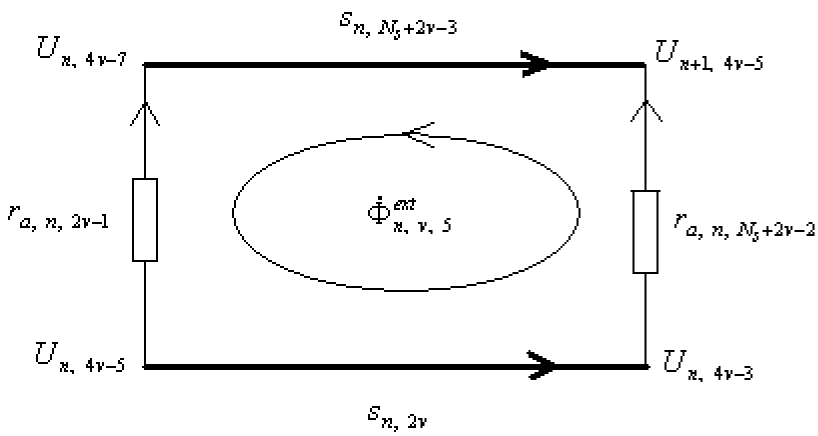

Figure 4(e) Face $f = 5$ of sub-networks $v = 2,\ldots,\ \frac{1}{2}N_s - 1$ in band n.

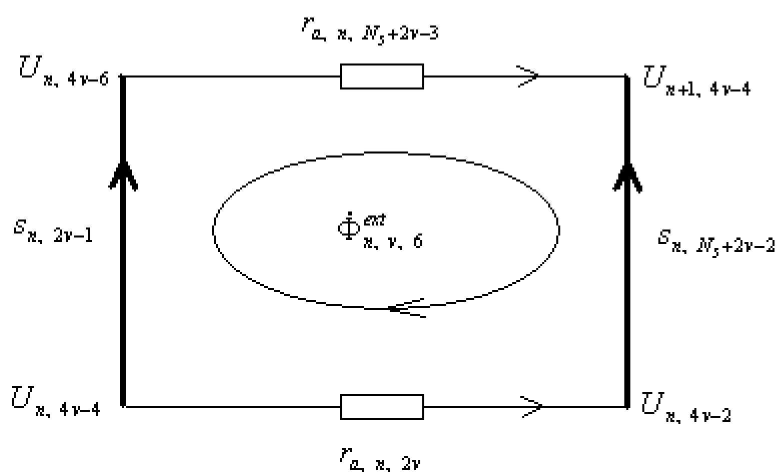

Figure 4(f) Face $f = 6$ of sub-networks $v = 2,\ldots,\ \frac{1}{2}N_s - 1$ in band n.

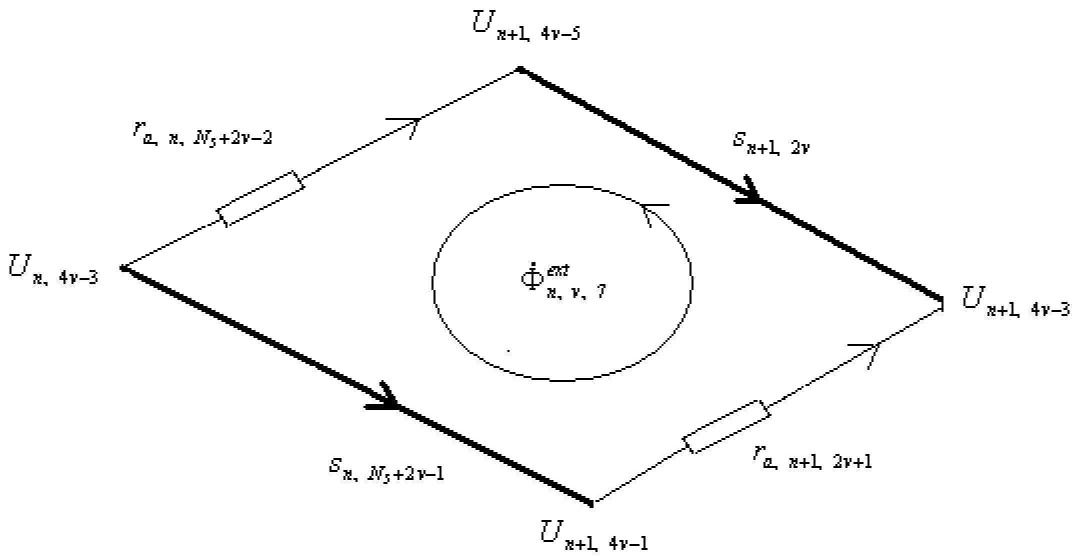

Figure 4(g) Face $f = 7$ of sub-networks $v = 2,\ldots,\ \frac{1}{2}N_s - 1$ in band n.

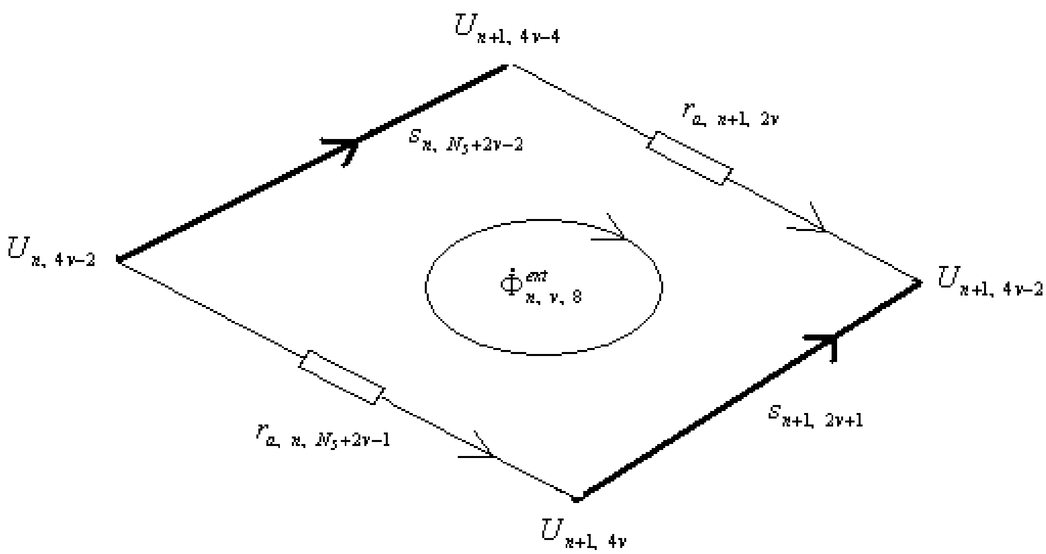

Figure 4(h) Face $f = 8$ of sub-networks $v = 2,\ldots,\ \frac{1}{2}N_s - 1$ in band n.

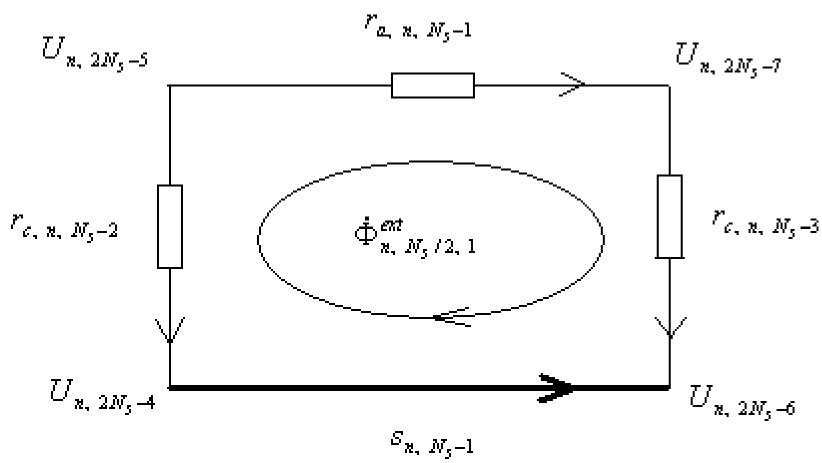

Figure 5(a) Face $f=1$ of sub-network $v=\frac{1}{2}N_s$ in band n.

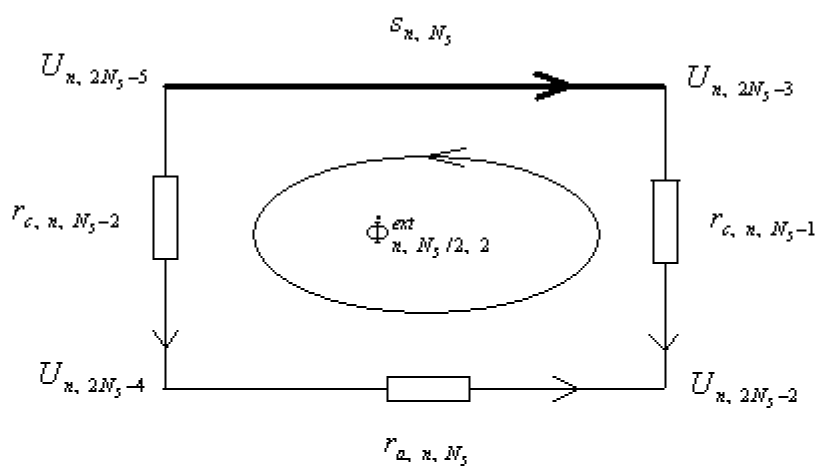

Figure 5(b) Face $f=2$ of sub-network $v=\frac{1}{2}N_s$ in band n.

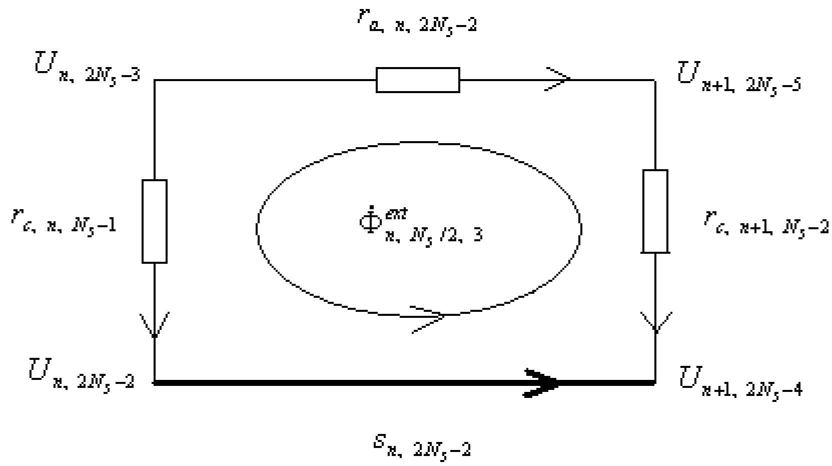

Figure 5(c) Face $f = 3$ of sub-network $v = \frac{1}{2}N_s$ in band n.

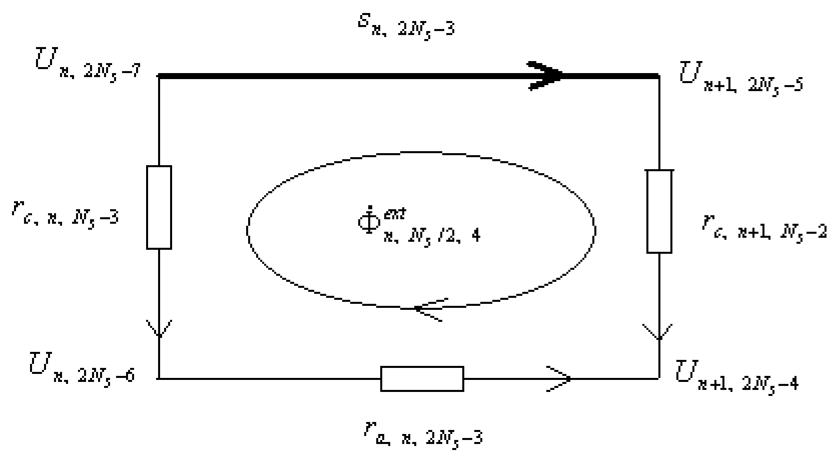

Figure 5(d) Face $f = 4$ of sub-network $v = \frac{1}{2}N_s$ in band n.

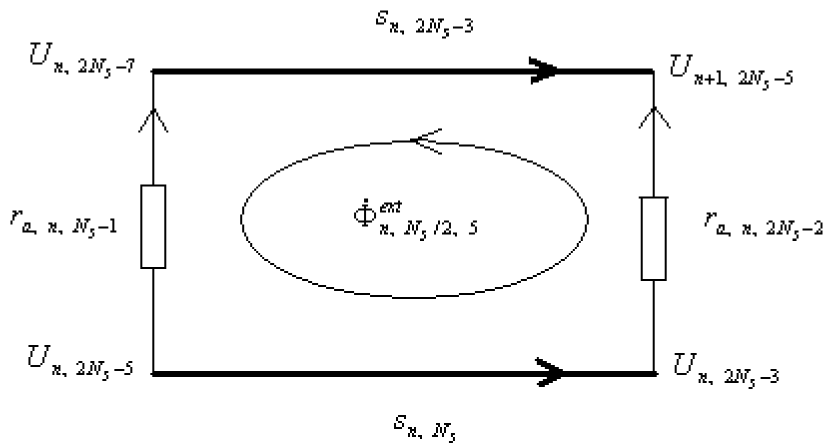

Figure 5(e) Face $f = 5$ of sub-network $v = \frac{1}{2} N_s$ in band n.

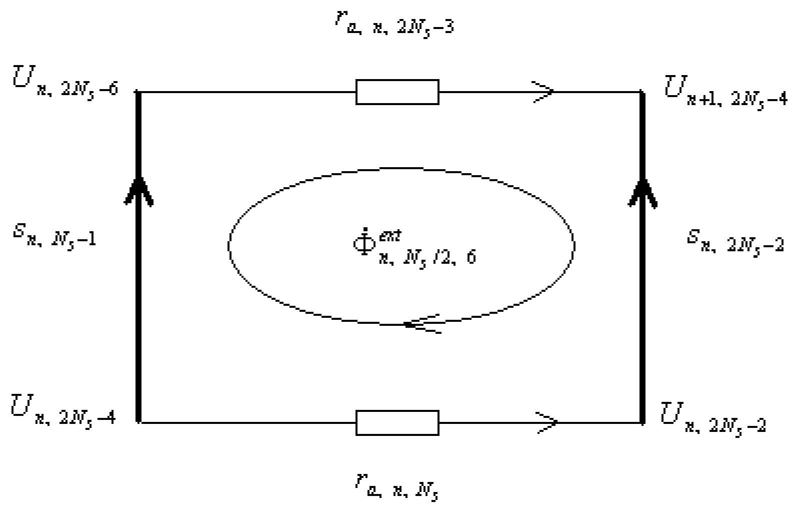

Figure 5(f) Face $f = 6$ of sub-network $v = \frac{1}{2} N_s$ in band n.

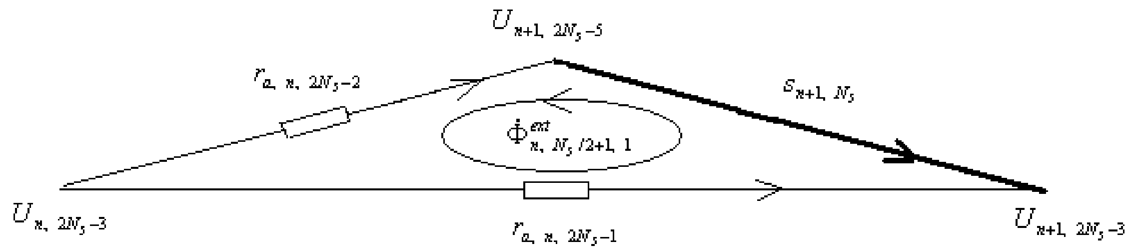

Figure 6(a) Face $f=1$ of sub-network $v=\frac{1}{2}N_s+1$ in band n.

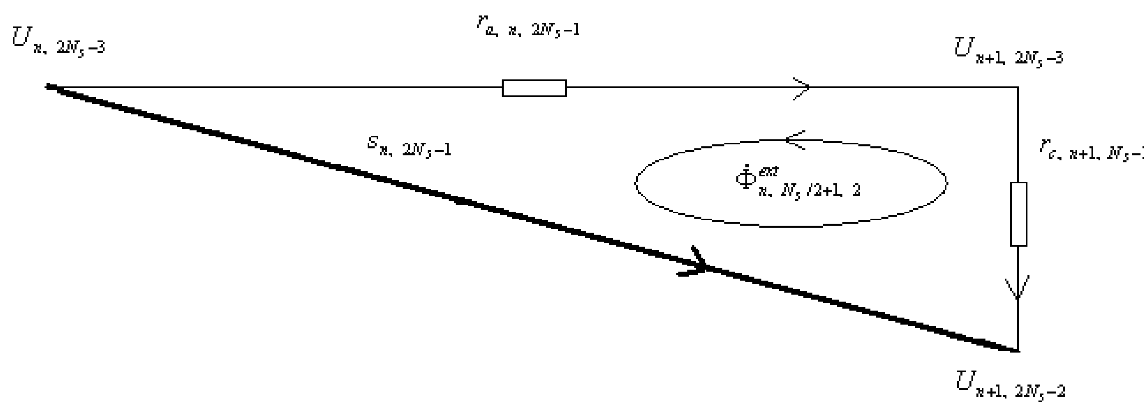

Figure 6(b) Face $f=2$ of sub-network $v=\frac{1}{2}N_s+1$ in band n.

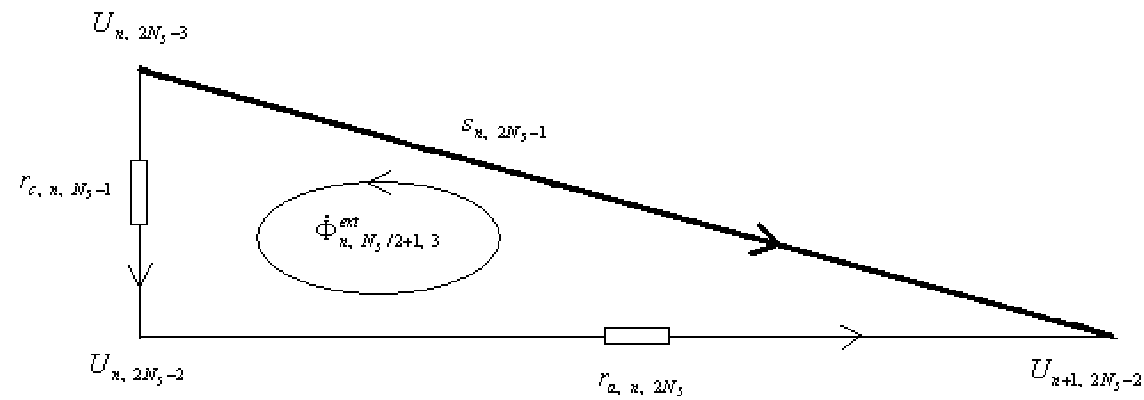

Figure 6(c) Face $f = 3$ of sub-network $v = \frac{1}{2}N_s + 1$ in band n.

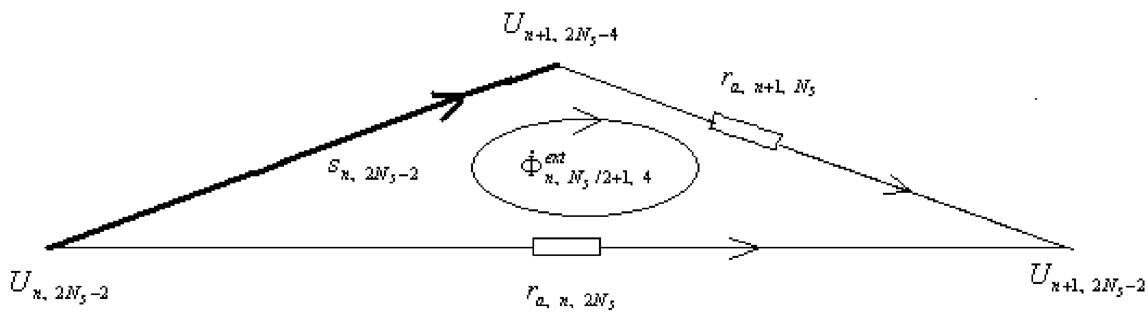

Figure 6(d) Face $f = 4$ of sub-network $v = \frac{1}{2}N_s + 1$ in band n.

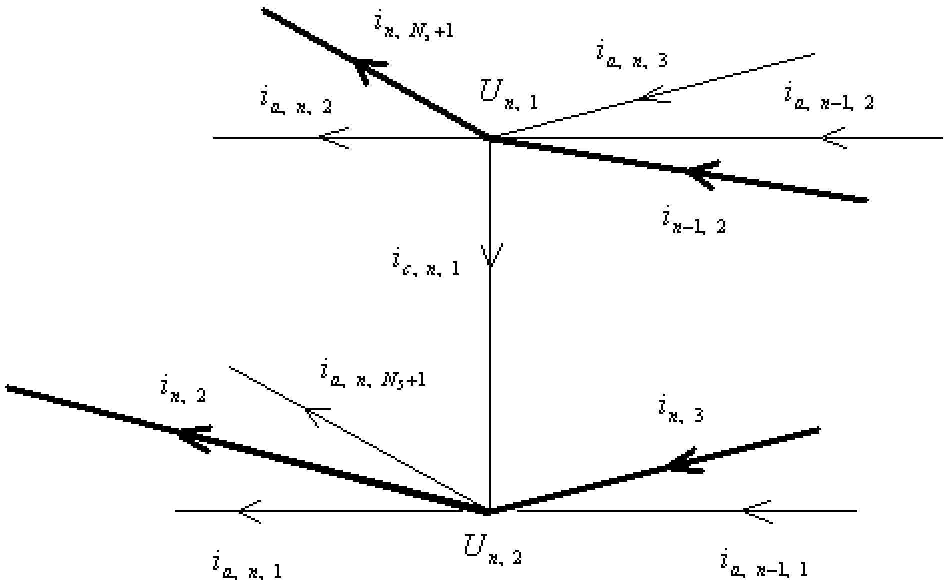

Figure 7(a) Nodes associated with sub-network $v = 1$ in band n.

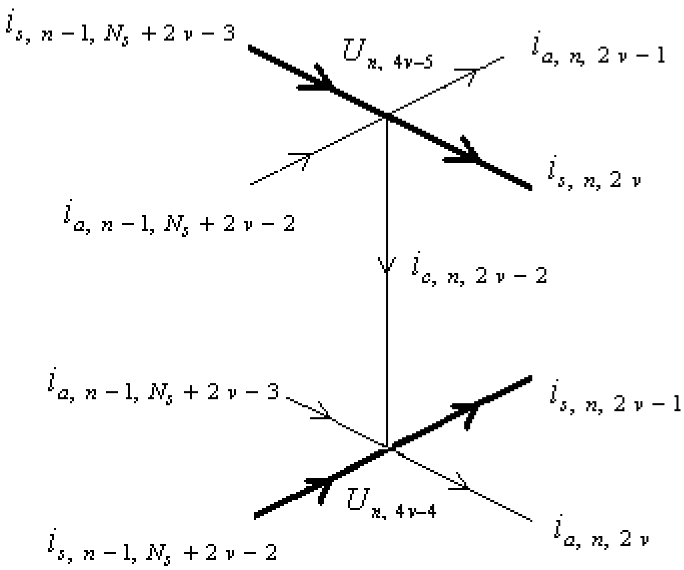

Figure 7(b) Nodes associated with sub-networks $v = 2, \ldots, \frac{1}{2}N_s$ in band n.

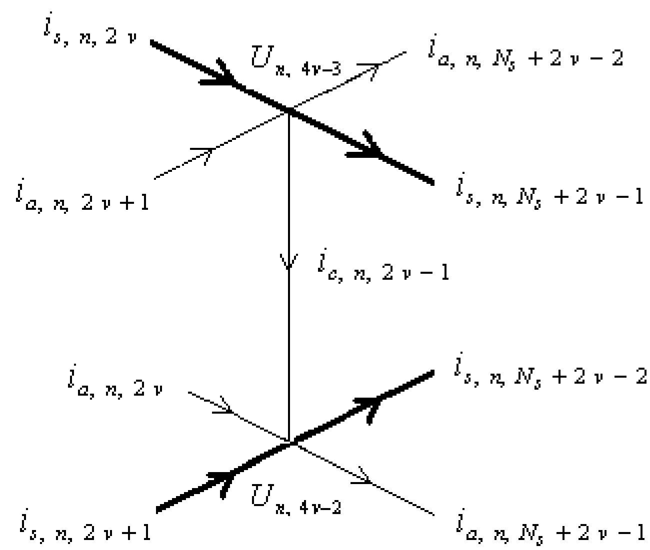

Figure 7(c) Nodes associated with sub-networks $v = 2, \ldots, \frac{1}{2}N_s - 1$ in band n.

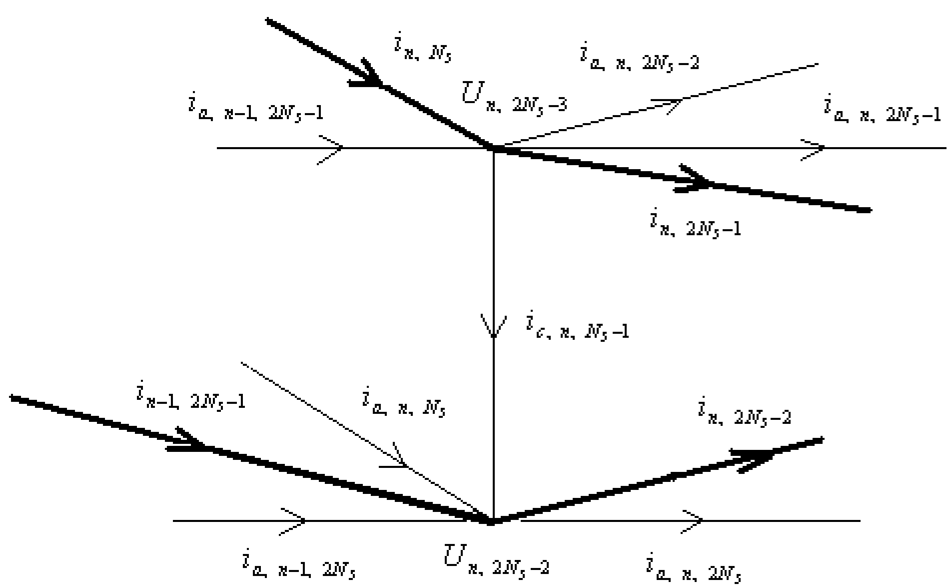

Figure 7(d) Nodes associated with sub-network $v = \frac{1}{2}N_s + 1$ in band n.